Zeros of linear combinations of Laguerre polynomials from different sequences

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Abstract

We study interlacing properties of the zeros of two types of linear combinations of Laguerre polynomials with different parameters, namely $R_n = L_n^\alpha + a L_n^{\alpha'}$ and $S_n = L_n^\alpha + b L_{n-1}^{\alpha'}$. Proofs and numerical counterexamples are given in situations where the zeros of R_n, and S_n, respectively, interlace (or do not in general) with the zeros of L_k^α, $L_k^{\alpha'}$, $k = n$ or $n - 1$. The results we prove hold for continuous, as well as integral, shifts of the parameter α.

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1 Introduction

Let μ be a positive Borel measure supported on a finite or infinite interval $[a, b]$ and let $\{p_n\}_{n=0}^{\infty}$ be the sequence of polynomials, uniquely determined up to normalization, orthogonal with respect to μ. Then $\int_a^b x^k p_n(x) d\mu(x) = 0$ for $k = 0, 1, \ldots, n-1$ and it is well known that the zeros of p_n are real and simple and lie in (a, b). Moreover, if $a < x_1 < x_2 < \ldots < x_n < b$ and $a < y_1 < y_2 < \ldots < y_{n-1} < b$ are the zeros of p_n and p_{n-1} respectively, then

$$a < x_1 < y_1 < x_2 < y_2 < \ldots < x_{n-1} < y_{n-1} < x_n < b,$$

a property usually called the interlacing of the zeros of p_n and p_{n-1}.

The interlacing of zeros of polynomials is particularly important in numerical quadrature (cf. [8]) and also arises, inter alia, in the context of extremal Zolotarev-Markov problems (cf. [5]), the Korous-Peebles problem (cf. [10]) and the Gelfond interpolation problem (cf. [4]).

In [6], Alfaro, Marcellán, Peña and Rezola derived necessary and sufficient conditions for the orthogonality of $\{Q_n\}_{n=0}^{\infty}$, where $Q_n(x) = p_n(x) + a_1 p_{n-1}(x) + \cdots + a_k p_{n-k}(x)$, $a_k \neq 0$, $n \geq k$ and $\{p_n\}_{n=0}^{\infty}$ is a sequence of monic orthogonal polynomials. Their work extends the results of Peherstorfer (cf. [7]) who established sufficient conditions, when $\text{supp}(\mu) = (-1, 1)$, on the real numbers $\{a_j\}_{j=1}^k$ such that $p_n + a_1 p_{n-1} + \ldots + a_k p_{n-k}$ has n simple zeros in $(-1, 1)$. Marcellán raised a more general question at OPSFA 2007: Given two different orthogonal sequences $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$, under what conditions does a linear combination $r_n = p_n + a q_n$, $a \neq 0$, form an orthogonal sequence $\{r_n\}_{n=0}^{\infty}$? A related question, relevant for applications, is whether and when the zeros of r_n interlace with the zeros of p_n, p_{n-1}, q_n or q_{n-1}. One starting point for answering these general questions is to consider linear combinations of classical orthogonal polynomials from different sequences but from the same family.

In this paper, we consider linear combinations of Laguerre polynomials L_n^{α} of the form $R_n^{\alpha,t} = L_n^{\alpha} + a \ L_n^{\alpha+t}$ and $S_n^{\alpha,t} = L_n^{\alpha} + b \ L_{n-1}^{\alpha+t}$ where $\alpha > -1$, $t > 0$ and $a, \ b \neq 0$. We recall that the Laguerre polynomials (cf. [9]) are orthogonal with respect to the weight function $e^{-x} x^{\alpha}$, $\alpha > -1$, on the interval $(0, \infty)$.

For $0 < t \leq 2$, we give proofs (or counterexamples) for the interlacing of the zeros of $R_n^{\alpha,t}$ and $S_n^{\alpha,t}$ with the zeros of L_n^{α}, $L_n^{\alpha+t}$, L_{n-1}^{α} and $L_{n-1}^{\alpha+t}$.

We will make use of two well known identities (cf. [1], 22.7.30 and 22.7.29)

$$L_n^\alpha = L_n^{\alpha+1} - L_{n-1}^{\alpha+1} \tag{1}$$
$$\text{and } xL_n^{\alpha+1} = (x-n)L_n^\alpha + (\alpha+n)L_{n-1}^\alpha \tag{2}$$

2 Linear combinations of Laguerre polynomials of the same degree

Let

$$R_n^{\alpha,t} = L_n^\alpha + aL_n^{\alpha+t}, \ a \neq 0, \ \alpha > -1. \tag{3}$$

Theorem 2.1 *For $0 < t \leq 2$, the zeros of $R_n^{\alpha,t}$ interlace with the zeros of (i)L_n^α, (ii) $L_n^{\alpha+t}$.*

Proof. We know from [3], Theorem 2.3 that the zeros of L_n^α interlace with the zeros of $L_n^{\alpha+t}$ for $0 < t \leq 2$ which implies that L_n^α has a different sign at successive zeros of $L_n^{\alpha+t}$ and vice versa. Evaluating (3) at succesive zeros x_i and x_{i+1} of L_n^α we obtain

$$R_n^{\alpha,t}(x_i)R_n^{\alpha,t}(x_{i+1}) = a^2 L_n^{\alpha+t}(x_i)L_n^{\alpha+t}(x_{i+1}), 1 = 1,2,\dots.n-1$$
$$< 0 \text{ for all } a \neq 0.$$

Therefore $R_n^{\alpha,t}$ has a different sign at successive zeros of L_n^α and so the zeros interlace. The same argument shows that the zeros of $R_n^{\alpha,t}$ interlace with those of $L_n^{\alpha+t}$ by evaluating (3) at successive zeros of $L_n^{\alpha+t}$. $\square$

Remark: For the integer values $t = 1$ and $t = 2$, $R_n^{\alpha,1}$ and $R_n^{\alpha,2}$ are in fact each a linear combination of orthogonal polynomials from the same sequence. Indeed, using the identity (1), we see that

$$R_n^{\alpha,1} = (a+1)L_n^{\alpha+1} - L_{n-1}^{\alpha+1}, \tag{4}$$

and the restrictions on a to ensure that $\{R_n^{\alpha,1}\}_{n=0}^\infty$ has all its zeros in $(0,\infty)$ can be deduced from ([2], Theorem 3(v)). Similarly, applying (1) iteratively, we obtain

$$R_n^{\alpha,2} = (a+1)L_n^{\alpha+2} - 2L_{n-1}^{\alpha+2} + L_{n-2}^{\alpha+2} \tag{5}$$

and the zeros of this type of linear combination are discussed in ([2], Theorem 5, and [8]).

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Evaluating (4) at successive zeros of $L_{n-1}^{\alpha+1}$, one can also prove that the zeros of $R_n^{\alpha,1}$ interlace with the zeros of $L_{n-1}^{\alpha+1}$. However, the zeros of $R_n^{\alpha,t}$ do not interlace with the zeros of L_{n-1}^{α} even in the simple special case when $t = 1$, as illustrated by the following example: For $n = 5$, $a = 2.33$, $\alpha = 1.45$ and $t = 1$, the zeros of $L_4^{1.45}$ are

$$\{0.954365,\ 2.94834,\ 6.26071,\ 11.6366\}$$

while those of $R_5^{1.45,1}$ are

$$\{1.17057,\ 3.01797,\ 5.80288,\ 9.83574,\ 15.9213\}.$$

3 Linear combinations of Laguerre polynomials of different degree

Next we consider linear combinations of the type

$$S_n^{\alpha,t} = L_n^{\alpha} + b\ L_{n-1}^{\alpha+t},\ \ b \neq 0,\ \alpha > -1. \tag{6}$$

We will need information on the interlacing properties of the two polynomials L_n^{α} and $L_{n-1}^{\alpha+t}$ in the linear combination.

Theorem 3.1 *Let $\alpha > -1$ and let*

$$
\begin{aligned}
0 < x_1 < x_2 < \ldots < x_n & \quad \textit{be the zeros of } \ L_n^{\alpha}, \\
0 < y_1 < y_2 < \cdots < y_{n-1} & \quad \textit{be the zeros of } \ L_{n-1}^{\alpha}, \\
0 < t_1 < t_2 < \ldots < t_{n-1} & \quad \textit{be the zeros of } \ L_{n-1}^{\alpha+t} \ \textit{and} \\
0 < X_1 < X_2 < \ldots < X_{n-1} & \quad \textit{be the zeros of } \ L_{n-1}^{\alpha+2}
\end{aligned}
$$

where $0 < t < 2$. Then

$$0 < x_1 < y_1 < t_1 < X_1 < x_2 < \ldots < x_{n-1} < y_{n-1} < t_{n-1} < X_{n-1} < x_n.$$

Proof. A simple computation using (1) and (2) leads to

$$(\alpha+1)L_n^{\alpha+1}(x) = (\alpha+n+1)L_n^{\alpha}(x) + xL_{n-1}^{\alpha+2}(x). \tag{7}$$

Evaluating (7) at successive zeros x_k and x_{k+1} of $L_n^{\alpha}(x)$, we obtain

$$x_k x_{k+1} L_{n-1}^{\alpha+2}(x_k) L_{n-1}^{\alpha+2}(x_{k+1}) \ = \ (\alpha+1)^2 L_n^{\alpha+1}(x_k) L_n^{\alpha+1}(x_{k+1}).$$

The expression on the right is negative since the zeros of L_n^α and $L_n^{\alpha+1}$ interlace (cf. [3], Theorem 2.3) and therefore

$$x_k < X_k < x_{k+1} \text{ for each fixed } k, \quad k = 1, \ldots, n-1.$$

The zeros of L_{n-1}^α increase as α increases (cf. [9], p. 122), hence

$$y_k < t_k < X_k \text{ for each fixed } k, \quad k = 1, \ldots, n.$$

Finally, since the zeros of L_n^α and L_{n-1}^α separate each other, we know that

$$x_k < y_k < x_{k+1} \text{ for each fixed } k, \quad k = 1, \ldots, n-1$$

and this completes the proof. □

Note that this result extends Theorem 2.4 in [3] to the case of polynomials of different degree with continuously varying parameters.

Theorem 3.2 *For $0 < t \leq 2$, the zeros of $S_n^{\alpha,t}$ interlace with the zeros of (i)L_n^α,   (ii) $L_{n-1}^{\alpha+t}$.*

Proof.   We know from Theorem 3.1 that the zeros of L_n^α interlace with the zeros of $L_{n-1}^{\alpha+t}$ for $0 < t \leq 2$ which implies that $L_{n-1}^{\alpha+t}$ has a different sign at successive zeros of L_n^α and vice versa. Evaluating (6) at successive zeros x_i and x_{i+1} of L_n^α we obtain

$$\begin{aligned} S_n^{\alpha,t}(x_i)S_n^{\alpha,t}(x_{i+1}) &= b^2 L_{n-1}^{\alpha+t}(x_i)L_{n-1}^{\alpha+t}(x_{i+1}) \, , i = 1, 2, \ldots .n-1 \\ &< \quad 0 \text{ for all } b \neq 0. \end{aligned}$$

Therefore $S_n^{\alpha,t}$ has a different sign at successive zeros of L_n^α and so the zeros interlace. The same argument shows that the zeros of $S_n^{\alpha,t}$ interlace with those of $L_{n-1}^{\alpha+t}$ by evaluating (6) at successive zeros of $L_{n-1}^{\alpha+t}$. □

It is interesting to note that in the case of linear combinations of Laguerre polynomials of different degree, the zeros of $S_n^{\alpha,t}$ do not interlace with the zeros of L_{n-1}^α. Indeed, even in the simplest case when $t = 1$ and $n = 5$, $b = 2.33$, $\alpha = 1.45$ in (6), the zeros of $S_5^{1.45,1}$ are

$$\{1.34638, \ 3.48132, \ 6.74108, \ 11.6384, \ 20.6928\}$$

while those of $L_4^{1.45}$ are

$$\{0.954365, \ 2.94834, \ 6.26071, \ 11.6366\},$$

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and interlacing does not occur. The zeros of $S_n^{\alpha,t}$ and $L_n^{\alpha+t}$ are interlacing when $t = 1$ since $S_n^{\alpha,1} = L_n^{\alpha+1} + (b-1)L_{n-1}^{\alpha+1}$. However, when $t = 2$, the zeros of $S_5^{1.45,2}$ are

$$\{1.94417,\ 4.47751,\ 8.08954,\ 12.6085,\ 16.7802\}$$

while those of $L_5^{1.45+2}$ are

$$\{1.70945,\ 3.92167,\ 7.07942,\ 11.5061,\ 18.0334\},$$

and interlacing fails in this case.

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